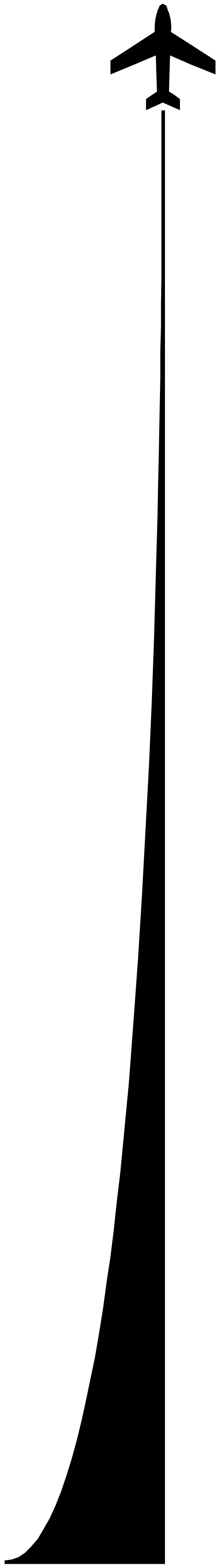

Session 6

Efficiency of Electronics Systems and Devices

ANALYSIS OF DIFFERENCE SCHEMES IN MODELING OF GYROTRON EQUATION

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Abstract

In this paper we will examine in detail different in the time two and three layer finite-difference schemes (Richardson's, Simpson's, Richtmayer-Morton's) that arises in solving single mode gyrotron equation. It is initial-boundary value problem for Schrödinger type partial differential equation with complex value boundary conditions of third kind. For stability analysis we will use the uniform grid in the space.

1. Formulation of Problem

We have the following initial-boundary value problem for Schrödinger type partial differential equation [1]

$$\begin{cases} i\dfrac{\partial f}{\partial t} = \dfrac{\partial^2 f}{\partial x^2} + \partial f + F \\ \quad f(t,0) = 0, \qquad \dfrac{\partial f(t,L)}{\partial x} = -i\gamma f(t,L),\ f(0,x) = f_0(x), \end{cases} \tag{1}$$

where $x \in (0,L)$ – is the space coordinate, $t > 0$ – is time, $\delta, \gamma > 0$ – are real parameters, $i = \sqrt{-1}$, $f(t,x)$ – is the unknown function, $f_0(x)$, $F(t,x)$ – are given complex functions.

Denote by $\omega_h = \{x_j = jh,\ j = \overline{1, N-1},\ Nh = L\}$ a uniform homogeneous spatial grid for *x* and by $\omega_\tau = \{t_n = n\tau,\ n-1,2,\ldots\}$ a temporal grid for *t* (*h* and τ are corresponding step-lengths). Substitute the continuous function $f = f(t,x)$ by the discrete grid function $y = y(t,x)$, $t \in \omega_\tau$, $x \in \omega_h$ with values $y(t_n, x_j) \equiv y_j^n$.

The derivatives of equation (1) approximate using the following finite-difference expressions:

1. the second order approximation in space

$$\frac{\partial^2 f(t_n, x_j)}{\partial x^2} = \Lambda f_j^n + r_1,$$

$$\Lambda f_j^n \equiv \frac{f(t_n, x_{j+1}) - 2f(t_n, x_j) + f(t_n, x_{j-1})}{h^2}$$

$$r_1 = -\frac{h^2}{12}\frac{\partial^4 f(t_n, \varsigma_1)}{\partial x^4},\ \varsigma_1 \in [x_{j-1}, x_{j+1}];$$

2. the first order approximation in time (for two level schemes)

$$\frac{\partial f\left(t_n,x_j\right)}{\partial t}=\frac{f\left(t_{n+1},x_j\right)-f\left(t_n,x_j\right)}{\tau}+r_2,$$

$$r_2-\frac{\tau}{2}\frac{\partial^2 f\left(\theta_2,x_j\right)}{\partial t^2},\ \theta_2\in\left[t_n,t_{n+1}\right];$$

3. the second order approximation in time (for three level schemes)

$$\frac{\partial f\left(t_n,x_j\right)}{\partial t}=\frac{f\left(t_{n+1},x_j\right)-f\left(t_{n-1},x_j\right)}{2\tau}+r_3,$$

$$r_3-\frac{\tau^2}{6}\frac{\partial^3 f\left(\theta_3,x_j\right)}{\partial t^3},\ \theta_3\in\left[t_{n-1},t_{n+1}\right].$$

The derivatives of boundary conditions (1) approximate using the following expressions:

1. the first order approximation in space

$$\frac{\partial f\left(t_n,L\right)}{\partial x}=\frac{f\left(t_n,x_N\right)-f\left(t_n,x_{N-1}\right)}{h}+r_4,$$

$$r_4=-\frac{h}{2}\frac{\partial^2 f\left(t_n,\varsigma_2\right)}{\partial x^2},\ \varsigma_2\in\left[x_{N-1},x_N\right],\ x_N=L,\ x_{N-1}=L-h;$$

2. the second order approximation in space

$$\frac{\partial f\left(t_n,L\right)}{\partial x}=\frac{3f\left(t_n,x_N\right)-4f\left(t_n,x_{N-1}\right)+f\left(t_n,x_{N-2}\right)}{2h}+r_5,$$

$$r_5=-\frac{h^2}{3}\frac{\partial^3 f\left(t_n,\varsigma_3\right)}{\partial x^3},\ \varsigma_3\in\left[x_{N-2},x_N\right],\ x_{N-2}=L-2h.$$

Hence, ignoring the remained terms r_4 and r_5, we have two approximation forms for boundary conditions in fixed time moment t_n:

$$y_N^n=Cy_{N-1}^n \tag{2}$$

$$y_N^n=\tilde{C}_1y_{N-1}^n+\tilde{C}_2y_{N-2}^n, \tag{3}$$

where $C=\left(1+i\gamma h\right)^{-1}$, $\tilde{C}_1=2\left(1.5+i\gamma h\right)^{-1}$, $\tilde{C}_2=-0.5\left(1.5+i\gamma h\right)^{-1}$.

Then ignoring the remained term r_1 we obtain 3-point difference expression, that approximate the second order derivative in the point $\left(t_n,x_j\right)$

$$\Lambda y_j^n\equiv\frac{\left(y_{j+1}^n-2y_j^n+y_{j-1}^n\right)}{h^2}. \tag{4}$$

To study the stability of the discrete problems (finite-difference schemes) we rewrite the homogeneous difference equations with respect to the difference $z_j^n=y_j^n-f\left(x_j,t_n\right)$ in the matrix operator form

$$z^{n+1}=Gz^n, \tag{5}$$

where G is the transition operator with the eigenvalues ρ_k and eigenfunctions $g^{(k)}\left(x_j\right)$, $k=\overline{1,N-1}$ of the difference operator Λ. The solution of difference equations corresponding to values z_j^n can be found in the form

$$z_j^n=\left(\rho_k\right)^n g^{(k)}\left(x_j\right),\ k=\overline{1,N-1}. \tag{6}$$

Using (2) the corresponding eigenvalues λ_k of difference operator Λ can be obtained from the three-point finite-difference scheme

$$\begin{cases}\Lambda g_j^{(k)}+\lambda_k g_j^{(k)}=0, & j=\overline{1,N-1}\\ g_0^{(k)}=0, & g_N^{(k)}=Cg_{N-1}^{(k)}\end{cases} \tag{7}$$

For the boundary condition (3) in (7) we introduce the expression $g_N^{(k)}=C_1 g_{N-1}^{(k)}+C_2 g_{N-1}^{(k)}$. Now the solution of (7) can be written as

$$g_j^{(k)}=E_k\sin\left(q_k x_j\right), \tag{8}$$

where E_k are arbitrary constants, $1-\frac{\lambda_k h^2}{2}=\cos\left(q_k h\right)$. From boundary conditions (2) follows that the complex parameter q_k is determined by the complex transcendental equation

$$\sin\left(q_k L\right)=C\sin\left(q_k\left(L-h\right)\right), \tag{9}$$

where the parameter $q_k=a_k+ib_k$, $k=\overline{1,N-1}$ has complex values. Therefore

$$\lambda_k=\frac{2}{h^2}\left(1-\cos\left(q_k h\right)\right)=A_k+iB_k, \tag{10}$$

where $A_k=2h^{-2}\left(1-\cos\left(a_k h\right)\cosh\left(b_k h\right)\right)$, $B_k=2h^{-2}\sin\left(a_k h\right)\sinh\left(b_k h\right)$, $k=\overline{1,N-1}$.

If $\gamma=\infty$ (boundary conditions of first kind $g_N^{(k)}=0$), then $C=0$ and $q_k=\frac{k\pi}{L}$ (real numbers),

$$\lambda_k=\frac{4}{h^2}\sin^2\frac{k\pi h}{2L} \tag{11}$$

and $g_j^{(k)}=\sqrt{\frac{2}{L}}\sin\frac{k\pi x_j}{2L}$, $k=\overline{1,N-1}$ [2].

Calculations with the help of MAPLE for $L=15$, $\gamma=2$, $h=0,1$ are shown in Table (1), where a_k and b_k are solutions of (9) and $\frac{\pi\left(k-1\right)}{L}<a_k<\frac{\pi k}{L}$, $0<b_k<1$.

Table 1. The discrete values of q_k , $\tilde{\alpha}_k$

k	a_k	b_k	A_k	B_k
1	0.2092	0.0070	0.0437	0.0029
2	0.4183	0.0141	0.1748	0.0118
3	0.6273	0.0216	0.3929	0.0271
4	0.8360	0.0296	0.6976	0.0494
5	1.0442	0.0383	1.0879	0.0798
6	1.2515	0.0481	1.5618	0.1201
7	1.4569	0.0595	2.1154	0.1728
8	1.6587	0.0725	2.7396	0.2393

In Fig. (1) we show the first 50 eigenvalues q_k for $h = 0,02$.

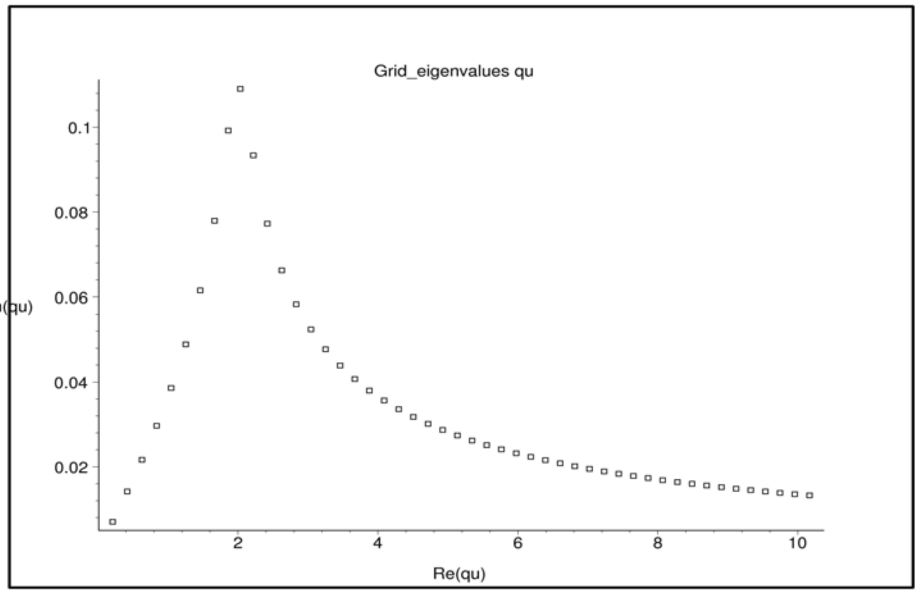

***Figure 1**. Eigenvalues of the discrete problem* q_k *,* $N = 750$

It is seen that, if $\left(a_k, b_k\right)$ is a solution of these systems, then also $\left(-a_k, -b_k\right)$ is a solution. The values of the coefficients A_k, B_k do not change and it is sufficient to consider only $a_k > 0$. If simultaneously $b_k > 0$, then also $B_k > 0$. Calculations with the help of "MAPLE" show that positive variables a_k correspond to positive variables b_k i.e. $B_k > 0$. If the parameter $\gamma < 0$, then it can be easily seen that positive a_k correspond to negative b_k .

Now we study in detail the stability and precision of different schemes for $\left|\rho_k\right| \le 1$, $k = \overline{1, N-1}$.

2. Two Level Finite-Difference Scheme with Parameter σ

Substituting differences (2) – (4) into the problem (1), we obtain a two-layer finite-difference scheme with weight $\sigma \in [0,1]$

$$\begin{cases} \dfrac{y_j^{n+1} - y_j^n}{\tau} = -i\left(\sigma\Lambda^* y_j^{n+1} + (1-\sigma)\Lambda^* y_j^n + F_j^\sigma\right), j = \overline{1, N-1} \\ y_0^{n+1} = 0,\ y_N^{n+1} = Cy_{N-1}^{n+1} \end{cases}, \tag{12}$$

where $F_j^\sigma = F\left(t_n + \sigma\tau, x_j\right)$, $\Lambda^* = \Lambda + \delta$. For the truncation error

$$\psi_j^n \equiv \frac{f_j^{n+1} - f_j^n}{\tau} + i\left(\sigma\Lambda^* f_j^{n+1} + (1-\sigma)\Lambda^* f_j^n + F_j^\sigma\right)$$

we get $\psi_j^n = O\left(\tau^\alpha + h^2\right)$, where $\alpha = 1$ if $\sigma \neq 0.5$ and $\alpha = 2$ if $\sigma = 0.5$.

Boundary conditions are approximated only to the first order. To obtain second order, one has use expression (3)

For error z_j^n (6) from (12) follows the expression

$$\rho_k = \frac{1 + i\tau(1-\sigma)\lambda_k^*}{1 - i\tau\sigma\lambda_k^*}, \tag{13}$$

where $\lambda_k^* = \lambda_k - \delta$.

If λ_k are real numbers, e.g., in the case of the first kind boundary conditions $\left(\gamma = \infty,\ z_N^{n+1} = 0\right)$ $q_k = \dfrac{k\pi}{L}$, then from the stability condition [2]

$$\left|\rho_k\right|^2 = \left(1 + \tau^2(1-\sigma)^2\left(\lambda_k^*\right)^2\right)\left(1 + \tau^2\sigma^2\left(\lambda_k^*\right)^2\right)^{-1} \leq 1,$$

it follows that

$$\sigma \geq \frac{1}{2} \tag{14}$$

independent of τ. Similar problem for Schrödinger type differential equation was investigated in [3].

Taking the boundary condition of the third kind in the form $z_N^{n+1} = Cz_{N-1}^{n+1}$ and determining the complex parameter $q_k = a_k + ib_k$, we find the complex values $\lambda_k^* = A_k^* + iB_k$, where $A_k^* = A_k - \delta$, $B_k > 0$.

Then from

$$\left|\rho_k\right|^2 = \frac{\left(1 - \tau(1-\sigma)B_k\right)^2 + \left(A_k^*\right)^2\tau^2(1-\sigma)^2}{\left(1 + \tau\sigma B_k\right)^2 + \left(A_k^*\right)^2\tau^2\sigma^2} \leq 1$$

follows that

$$-2B_k + \tau(1 - 2\sigma)\left(\left(A_k^*\right)^2 + B_k^2\right) \leq 0,$$

the inequality (14) holds and the difference scheme (12) is stable.

3. Richardson's Three Level Finite-Difference Scheme

The difference equations of three level Richardson's scheme are in the following form

$$\frac{y_j^{n+1} - y_j^{n-1}}{2\tau} = -i\left(\Lambda * y_j^n + F_j^n\right). \qquad (15)$$

Corresponding truncation error is $\psi_j^n = O\left(\tau^2 + h^2\right)$. For real heat transfer equations ($-i$ is replaced by 1) this scheme is absolutely unstable [2]. The transfer modulus ρ_k can be obtained from the following quadratic equation

$$\rho_k^2 - 2i\tau\lambda_k^* \rho_k - 1 = 0 . \qquad (16)$$

For the stability of the difference scheme it is necessary that the all roots of (16) satisfy the inequality $|\rho_k| \le 1$ (the identity is valid only for the simple roots of (16)). For the complex quadratic equation

$$a\rho^2 + b\rho + c = 0 \qquad (17)$$

this inequalities holds, if [4]

$$\left|\bar{b}a - \bar{c}b\right| \le |a|^2 - |c|^2 , \ |b| < 2|a| , \qquad (18)$$

where $\bar{b}$, $\bar{c}$ are the complex conjugate values of b, c .

If $\lambda_k^* \in R$, then from (18) follows that

$$\tau\left|\lambda_k^*\right| < 1, \quad \tau < \frac{h^2}{4 - h^2\delta} . \qquad (19)$$

If $\lambda_k^* \in C$, then from (18) follows that

$$B_k = 0, \quad \tau < \frac{1}{\max\left|A_k^*\right|}, \qquad (20)$$

which is equivalent to (19).

Therefore, if $B_k > 0$, then Richardson's scheme for equation (1) is absolutely unstable and conditionally stable by (19) only in the limit case $\gamma = \infty$ $\left(B_k = 0\right)$ for the boundary conditions of the first kind.

4. Simpson's Three Level Finite-Difference Scheme

The difference equations of three level Simpson's scheme is in the following form

$$\frac{y_j^{n+1} - y_j^{n-1}}{2\tau} = -i\left(\left(\Lambda * y_j^{n+1} + 4\Lambda * y_j^n + \Lambda * y_j^{n-1}\right)/6 + F_j^n\right). \qquad (21)$$

The truncation error is $\psi_j^n = O\left(\tau^4 + h^2\right)$. For real heat transfer equations this scheme is absolutely unstable [2]. The transfer modulus ρ_k can be obtained from the following quadratic equation

$$\left(1-\frac{i\tau\lambda_k^*}{3}\right)\rho_k^2-\frac{4i\tau\lambda_k^*}{3}\rho_k-\left(1+\frac{i\tau\lambda_k^*}{3}\right)=0. \qquad (22)$$

If $\lambda_k^* \in R$, then from (18) follows that

$$\left(\tau\lambda_k^*\right)^2<3,\quad \tau<\sqrt{3}\frac{h^2}{4-h^2\delta}. \qquad (23)$$

If $\lambda_k^* \in C$, then from (18) follows that

$$B_k=0,\quad \tau<\frac{\sqrt{3}}{\max\left|A_k^*\right|}, \qquad (24)$$

which is equivalent to (23).

Therefore, if $B_k>0$, then Simpson's scheme also for equation (1) is absolutely unstable and conditionally stable by (23)) only if $\gamma=\infty$ $\left(B_k=0\right)$ for the boundary conditions of the first kind.

5. Richtmayer-Morton's Three Level Finite-Difference Scheme

The difference equations of three level Richtmayer-Morton's scheme is in the following form

$$1.5\frac{y_j^{n+1}-y_j^n}{\tau}-0.5\frac{y_j^n-y_j^{n-1}}{\tau}=-i\left(\Lambda * y_j^{n+1}+F_j^{n+1}\right). \qquad (25)$$

The truncation error is $\psi_j^{n+1}=\tau^2+h^2$. For real heat transfer equations this scheme is absolutely stable [2]. The transfer modulus ρ_k can be obtained from the following quadratic equation

$$\left(1.5-i\tau\lambda_k^*\right)\rho_k^2+2\rho_k+0.5=0. \qquad (26)$$

If $\lambda_k^* \in R$, then from (18) follows that this difference scheme is absolutely stable $\left(\tau<\infty\right)$.

If $\lambda_k^* \in C$, then from (18) follows also that this difference scheme is absolutely stable, if $B_k>0$.

6. General Three Level Finite-Difference Schemes

For fixed grid point x_j in the space we can considered 1-D three level approximation in the time of the function $u=u\left(t\right)$ in following form:

$$\frac{Au\left(t_{n+1}\right)+Bu\left(t_n\right)+Cu\left(t_{n-1}\right)}{\tau}=A_1u'\left(t_{n+1}\right)+B_1u'\left(t_n\right)+C_1u'\left(t_{n-1}\right)+\tau_n, \qquad (27)$$

where $\tau_n=\tau^{\alpha-1}\frac{u^{(\alpha)}\left(\varsigma_n\right)}{\alpha!}C_0$ is the error term, $\varsigma\in\left[t_{n-1},t_{n+1}\right]$, A,B,C,A_1,B_1,C_1,C_0, $\alpha\geq 3$ are the unknown coefficients, $u'=\frac{d}{dt}$, $u'\left(t_n\right)=-i\left(\Lambda * y_j^n+F_j^n\right)$.

The unknown coefficients can be obtained from the condition, that the expression (27) is valid for polynomials with highest degree. To this end we consider normalized function $\tilde{u}(\tilde{t})$, where $\tilde{t} = \frac{t - t_n}{\tau}$. Then from (27) follows the expression

$$A\tilde{u}(1) + B\tilde{u}(0) + C\tilde{u}(-1) = A_1\tilde{u}'(1) + B_1\tilde{u}'(0) + C_1\tilde{u}'(-1) + C_0 \frac{\tilde{u}^{(\alpha)}(\tilde{\varsigma})}{\alpha!} \quad (28)$$

where $\tilde{u}' = \frac{d\tilde{u}}{d\tilde{t}}$, $\tilde{\varsigma} \in [-1,1]$. Using the elementary functions $\tilde{u}(\ddot{t}) = \tilde{t}^k$, $k = \overline{0,\alpha}$, we obtain the unknown coefficients from the following system of linear algebraic equations:

$$A + B0^k + C(-1)^k = k\left(A_1 + B_1 0^{k-1} + C_1(-1)^{k-1}\right), \quad (29)$$

where $0^0 = 1$, $k = \overline{0,\alpha - 1}$.

If $k = \alpha$, then we obtain the equation for coefficient C_0

$$C_0 = A + C(-1)^a - \alpha\left(A_1 + C_1(-1)^{\alpha-1}\right). \quad (30)$$

For $\alpha = 5$ from (29), (30) can be obtained the Simpson finite difference equations (21) with

$$A = \frac{1}{2},\ B = 0,\ C = -\frac{1}{2},\ A_1 = \frac{1}{6},\ B_1 = \frac{4}{6},\ C_1 = \frac{1}{6},\ C_0 = -\frac{4}{6}.$$

For $\alpha = 3$ can be obtained the Richtmayer-Morton finite difference equations (25) with

$$A = \frac{3}{2},\ B = -2,\ C = \frac{1}{2},\ A_1 = 1,\ B_1 = C_1 = 0,\ C_0 = -2.$$

For $\alpha = 4$ can be obtained the finite difference equations, depending of two parameters A, C, in the form

$$\begin{cases} B = -(A + C),\ A_1 = \frac{1}{12}(5A + C),\ B_1 = \frac{2}{3}(A - C), \\ C_1 = -\frac{1}{12}(A + 5C),\ C_0 = -(A + C) \end{cases}. \quad (31)$$

If $C = -A$, then we obtain (21).

The stability investigations of general difference equations (27) $(r_n = 0)$ lead to the following characteristic equations

$$(A - \mu_k A_1)\rho_k^2 + (B - \mu_k B_1)\rho_k + (C - \mu_k C_1) = 0,$$

where $\mu_k = i\tau\lambda_k^*$.

From (18) follows inequalities:

1) if $\lambda_k^* \in R$, then

$$\begin{cases} \left|(B + \mu_k B_1)(A - \mu_k A_1) - (C + \mu_k C_1)(B - \mu_k B_1)\right| \le \\ \qquad A^2 - (\mu_k A_1)^2 - C^2 + (\mu_k C_1)^2, \\ \qquad B^2 - (\mu_k B_1)^2 < 4\left(A^2 - (\mu_k A_1)^2\right); \end{cases} \quad (32)$$

2) if $\lambda_k^* \in C$, then $\left(\mu_k^* = i\tau A_k^*\right)$

$$\begin{cases} \left|\left(B + \mu_k^* B_1 + \tau B_1 B_k\right)\left(A - \mu_k^* A_1 + \tau A_1 B_k\right) - \left(C + \mu_k^* C_1 + \tau C_1 B_k\right)\left(B - \mu_k^* B_1 + \tau B_1 B_k\right)\right| \le \\ \qquad \left(A + \tau A_1 B_k\right)^2 - \left(\mu_k^* A_1\right)^2 - \left(C + \tau C_1 B_k\right)^2 + \left(\mu_k^* C_1\right)^2, \\ \qquad \left(B + \tau A_1 B_k\right)^2 - \left(\mu_k^* B_1\right)^2 < 4\left(\left(A + \tau A_1 B_k\right)^2 - \left(\mu_k^* A_1\right)^2\right). \end{cases} \tag{33}$$

From (31) by $A = C = 1$ it follows, that

$$A = 1,\ C_1 = -\frac{1}{2},\ A_1 = \frac{1}{2},\ C = 1,\ B_1 = 0,\ B = -2,\ C_0 = -2,$$

and inequalities (32), (33) holds.

Therefore we obtain the following three level finite difference equations

$$\frac{y_j^{n+1} - y_j^n}{\tau} - \frac{y_j^n - y_j^{n-1}}{\tau} = -i\left(0.5\left(\Lambda^* y_j^{n+1} - \Lambda^* y_j^{n-1} + F_j^{n+1} - F_j^{n-1}\right)\right) \tag{34}$$

with the truncation error

$$\psi_j^n = -\frac{\tau^3}{12}\frac{\partial^4 f\left(\theta_n, x_j\right)}{\partial t^4} + \frac{ih^2}{12}\frac{\partial^4 f\left(t_n, \varsigma_j\right)}{\partial x^4},$$

where $\varsigma_j \in \left[x_{j-1}, x_{j+1}\right]$, $\theta_n \in \left[t_n, t_{t+1}\right]$.

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HIGH-SELECTIVE DIGITAL FILTERS WITH OPTIMIZED LINEARITY OF PHASE CHARACTERISTIC

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The method of increase of phase characteristic linearity of high-selective filters is offered. The modelling results are introduced.

Key words: transfer function with complex coefficients, linear phase characteristic

One of the important additional requirements at designing supernarrow-band filters frequently is a high linearity of phase-frequency characteristic. Especially sharply this problem arises at designing high-selective filters. Unfortunately, the regular approach at definition of the best realization meeting these requirements does not exist yet; therefore the problem is reduced to optimization selection its parameters at imitating or natural modelling [1].

In the work the method of filters' phase characteristics linearization by a choice of parameters of transfer function with complex coefficients is offered.

As base, we shall consider a transfer function of the prototype of the elliptic filter possessing the maximal selectivity with other equal conditions:

$$H(p) = H_0 \prod_{i=1}^{N/2} \frac{p^2 + A_i}{p^2 + B_i p + C_i} \quad \text{- odd order} \tag{1}$$

$$H(p) = \frac{H_0}{p+G} \prod_{i=1}^{N-1/2} \frac{p^2 + A_i}{p^2 + B_i p + C_i} \quad \text{- even order} \tag{2}$$

Let $p = k\frac{1-z^{-1}}{1+z^{-1}}$, so transfer function of separate biquad in Z-plane:

$$H_i(z^{-1}) = h_{0i} \frac{(k^2 + A_i) + 2(A_i - k^2)z^{-1} + (k^2 + A_i)z^{-2}}{1 + 2h_{0i}(C_i - k^2)z^{-1} + h_{0i}(k^2 - B_i k + C_i)z^{-2}}, \tag{3}$$

where $h_{0i} = \frac{1}{k^2 + B_i k + C_i}$, $k = ctg\frac{\pi}{2}\tilde{\Omega}_1$.

The coefficients of transfer function are real. Let's consider, for example, frequency characteristics of the elliptic filter at the following requirements:

1. cut-off frequency, $\tilde{\Omega}_1 = 0.2$;
2. passband attenuation, a_{max} = 0.1 dB;
3. stopband attenuation, a_{min} = 60 dB.

These requirements are carried out by the filter of 6-th order.

The result is shown on picture 1. Thus the maximal deviation of a phase in a passband reaches size about 40-50 grad.

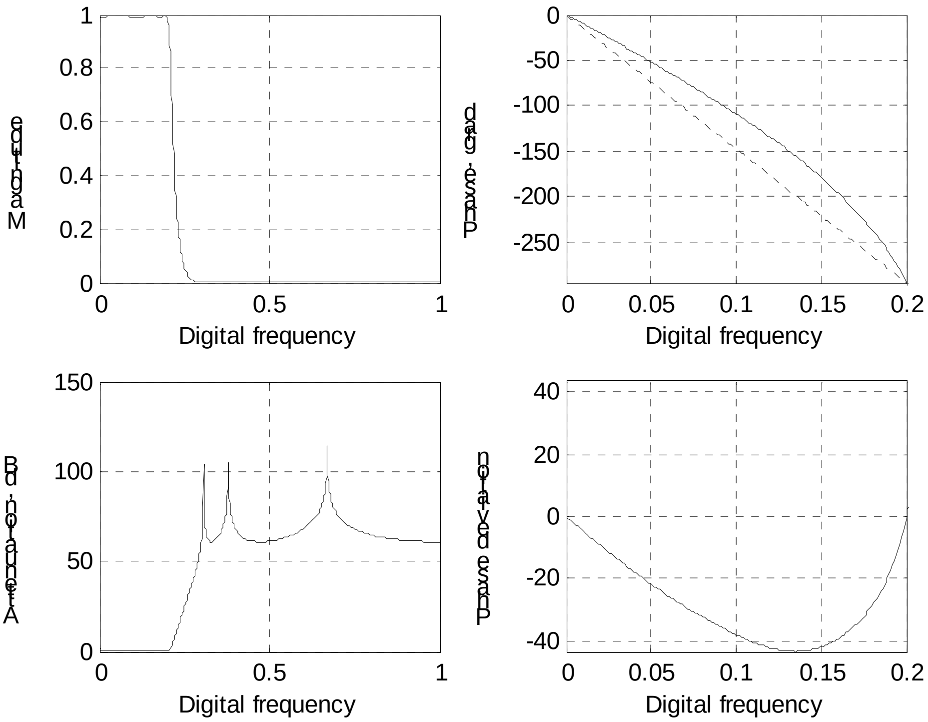

Figure 1. *Frequency characteristics of elliptic filter*

The absence of analytical methods for designing supernarrow-band digital filters with a linear phase forces to address to numerical methods of the decision.

As the possible approach to phase linearization, instead of traditional transfer function (3) it is offered to use the modified function:

$$H_i(z^{-1}) = \dot{h}_{0i}\,\frac{(k_1^{\,2} + A_i + \alpha_1 + j\beta_1) + 2(A_i - k_1^{\,2} + \alpha_2 + j\beta_2)z^{-1} + (k_1^{\,2} + A_i + \alpha_3 + j\beta_3)z^{-2}}{1 + 2\dot{h}_{0i}(C - k_1^{\,2} + \alpha_5 + j\beta_5)z^{-1} + \dot{h}_{0i}(k_1^{\,2} - Bk_1 + C + \alpha_6 + j\beta_6)z^{-2}} \qquad (4)$$

where $\dot{h}_{0i} = h_{0i} + (\alpha_4 + j\beta_4)$, $k_1 = k + \alpha_7$.

The parameter k_1 defines the cut-off frequency of modified amplitude-frequency characteristic. It’s possible to show, that at minimization of phase deviation in a passband the cut-off frequency increases.

All multipliers of modified biquad generally are complex. It complicates the realization; however there are approaches to the decision of this problem [2].

The analysis of the synthesized filter we shall lead as follows:

1. We shall weaken requirements to cut-off frequency, control frequency and passband attenuation of the modelled filter, keeping the order.
2. We shall set the requirement to the maximal deviation of a phase in a passband (at least, much less to traditional deviation).
3. Selecting coefficients in (4), we optimize a phase a numerical method (for example, Nelder-Mead), keeping the control above amplitude-frequency characteristic.
4. If the received maximal deviation of a phase in a passband does not meet the requirement, should set other initial approach and repeat item 3-4.

As a result of iterative optimization the following result is received: deviation of a phase in a passband - about 10^{-1}-10^{-2} grad. Thus the resulting mistake depends on rigidity of requirements to control frequency. Varying control frequency in allowable limits, it is possible to reduce considerably phase deviation in a passband.

On fig. 2 frequency characteristics after two iterations of global search are shown.

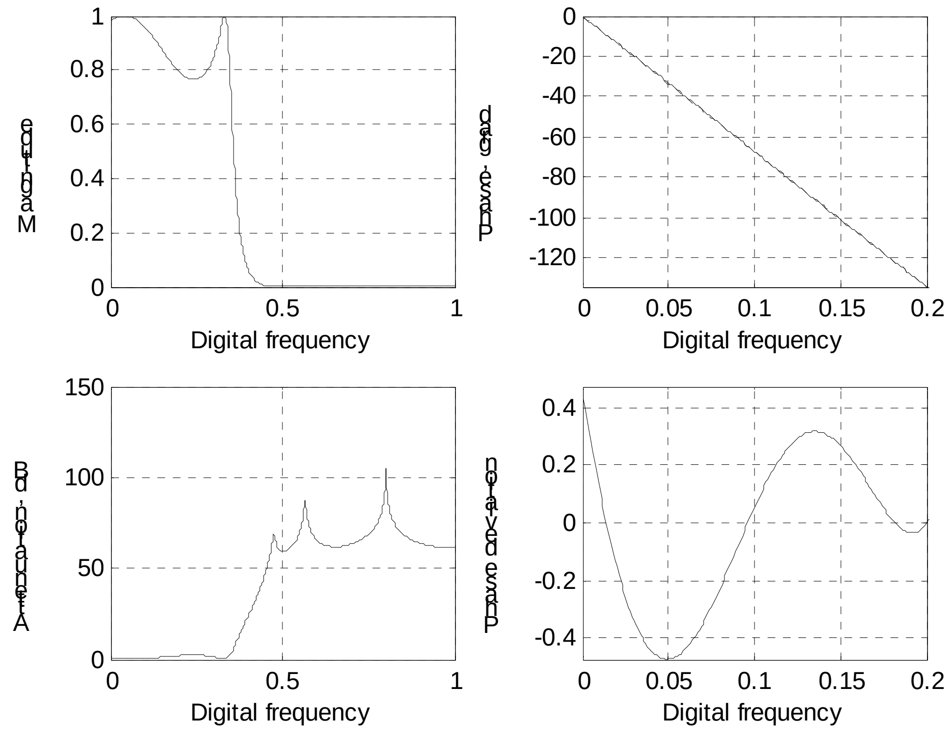

Figure 2. *Frequency characteristics of synthesized filter*

At optimization of a phase deviation in a working band it is necessary to supervise required attenuation on control frequency and higher. Thus cut-off frequency can be changed in limits in which safety of the order is guaranteed.

Conclusion

One of possible approaches to a problem of increase of phase characteristic linearity of high-selective filters at insignificant easing requirements to other parameters is offered.

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