SYNTHESIS OF COMPUTATIONAL-EFFECTIVE STRUCTURES OF DIGITAL FILTERS

Vitaly Yeremeyev, Takhir Mamirov

Transport and Telecommunication Institute
Lomonosov Str. 1, Riga, LV-1019, Latvia
Ph. (+371)29499489. Fax: (+371)7100660. E-mail: mamirov_t@yahoo.com

The new technique of digital filters synthesis on computational-effective biquad structures is offered. It is based on the developed special Z-transformation of transfer function of analogue polynomial prototype, when the structure of the designed digital filter has only a recursive part. This method allows simplifying the filter's biquades almost twice, thus amplitude-frequency responses of narrow-band filters practically coincide with traditional ones.
The examples of low-, high-frequency and passband filters on modified biquades and bilines are shown. Characteristics of new structures were compared with already known ones.

Keywords: *digital filters, computational-effective structures*

1. INTRODUCTION

Traditional structures of digital filters on biquades basically possess undesirable redundancy because of presence both recursive and non-recursive branches. This redundancy allows to synthesize digital filters with stable required characteristics independently either they narrowband or wideband.

But in particular cases there is a possibility to research and pay attention to only definite areas of frequency range, thus it's possible to simplify the structure of digital filter considerably. In this case the new class of polynomial characteristics may be discussed. Below the approach of this class synthesis is researched and the advantages before traditional approach are shown.

2. DESCRIPTION OF RESEARCH. LOW AND HIGH-FREQUENCY MODIFIED STRUCTURES

In the article [1] the examples of traditional biquades and bilines synthesis are resulted. Such structures possess undesirable structural redundancy, that is explained by presence both recursive and not recursive parts.

The description of a technique of computational-effective structures synthesis of digital filters is resulted. These structures are described by a new class of characteristics, namely polynomial transfer functions:

$$H(p) = \prod_{i=1}^{N/2} \frac{b_i}{p^2 + a_i p + b_i} = \prod_{i=1}^{N/2} \frac{b_i}{(p - p_{1i})(p - p_{2i})} . \qquad (1)$$

Let's consider 4 known classes of the prototypes widely used in DSP:

1. Butterworth;
2. Chebyshev's type I;
3. Chebyshev's type II;
4. Elliptical.

Elliptical filters and Chebyshev's type II filters are described by is fractional-rational transfer functions, that obviously assumes presence of zero of transfer. Chebyshev's type I prototypes possess the best selectivity among filters with monotonous amplitude-frequency response in a stopband, therefore they have been chosen as base. Generally for the odd order transfer function of Chebyshev's type I filter is described by the expression (2) or (3).

$$H(p)=\frac{1}{\gamma+p}\prod_{i=1}^{N-1/2}\frac{\gamma^2+\cos^2\left(\frac{2i-1}{2N}\pi\right)}{p^2+2\gamma\sin\left(\frac{2i-1}{2N}\pi\right)p+\gamma^2+\cos^2\left(\frac{2i-1}{2N}\pi\right)}=$$

$$=\frac{1}{\gamma+p}\prod_{i=1}^{N-1/2}\frac{b_i}{p^2+a_i p+b_i}. \qquad (2)$$

$$H(p)=\frac{1}{\gamma+p}\prod_{i=1}^{N-1/2}\frac{b}{(p-p_{i,1})(p-\overset{*}{p}_{i,1})}. \qquad (3)$$

In digital area transfer function of Chebyshev's type I filter is described by the expression:

$$H(z^{-1}) = \frac{b_0+b_1 z^{-1}}{1+a_1 z^{-1}}\prod_{i=1}^{N-1/2}\frac{b_{0i}+b_{1i} z^{-1}+b_{2i} z^{-2}}{1+a_{1i} z^{-1}+a_{2i} z^{-2}}. \qquad (4)$$

The low-frequency Z-transformation, which leads to this fractional-rational transfer function, uses the following frequency transformation:

$$\Omega = k\cdot\tan\left(\frac{\pi}{2}\tilde{\Omega}\right). \qquad (5)$$

then

$$\Omega = k\frac{\sin(\pi/2\,\tilde{\Omega})}{\cos(\pi/2\,\tilde{\Omega})} = k\frac{(1-\cos(\pi\tilde{\Omega}))^{1/2}}{(1+\cos(\pi\tilde{\Omega}))^{1/2}} = k\frac{\left(1-\frac{z+z^{-1}}{2}\right)^{1/2}}{\left(1+\frac{z+z^{-1}}{2}\right)^{1/2}} = k\left(\frac{2z^{-1}-1-z^{-2}}{2z^{-1}+1+z^{-2}}\right)^{1/2}. \qquad (6)$$

$$p = j\Omega \quad\Rightarrow\quad p = k\frac{1-z^{-1}}{1+z^{-1}}. \qquad (7)$$

It is offered to use the ***modified*** frequency Z-transformation which gives the best approximation in the area of the low frequencies. This is a sinus transformation, described by the expression (8):

$$\Omega = k\cdot\sin\left(\frac{\pi}{2}\tilde{\Omega}\right). \qquad (8)$$

The given transformation leads to the following transformation of the prototype poles:

$$\Omega^2 = k^2 \sin^2 (\pi/2\,\tilde{\Omega}) = \frac{k^2}{2}(1-\cos(\pi\tilde{\Omega})) = \frac{k^2}{2}\left(1-\frac{z+z^{-1}}{2}\right) = \frac{k^2}{4}\left(\frac{2z^{-1}-1-z^{-2}}{z^{-1}}\right), \quad (9)$$

$$p^2 = -\Omega^2 \qquad\qquad k^2 z^{-2} - (2k^2 + 4p^2)z^{-1} + k^2 = 0\,, \quad (10)$$

$$\left(z^{-1}\right)_{1,2} = 1 + 2\left(\frac{p}{k}\right)^2 \pm \sqrt{\left(1+2\left(\frac{p}{k}\right)^2\right)^2 - 1}\,. \quad (11)$$

It is possible to show graphically, that in low-frequency area both transformations are practically identical to each other (Fig. 1, 2).

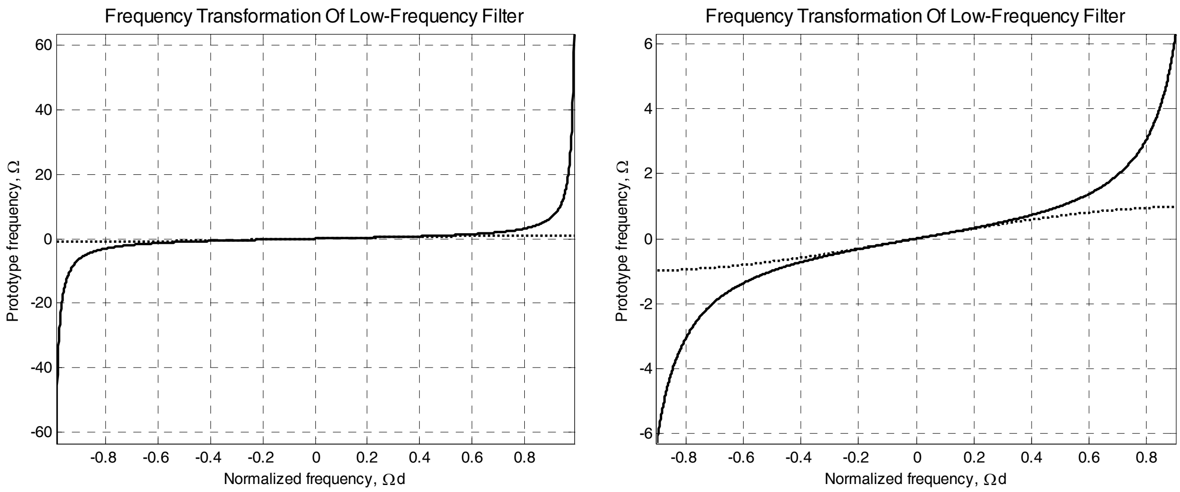


Figure 1. *Graphical representation of low-frequency transformation*

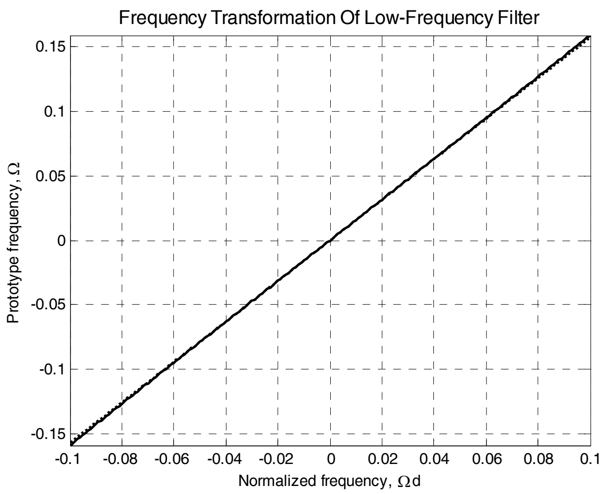


Figure 2. *Graphical representation of low-frequency transformation (zoomed)*

It is visible in this case, that in the field of digital frequencies [0:0.1] sinus transformation with very small mistake quite replaces tangent. The digital filter with polynomial transfer function will be described as follows:

$$H(z^{-1}) = \frac{1}{(\gamma+k)+(\gamma-k)z^{-1}} \prod_{i=1}^{N-1/2} \frac{b_i}{(1-z_i \cdot z^{-1})(1-\overset{*}{z_i} \cdot z^{-1})}, \qquad (12)$$

$$H(z^{-1}) = \frac{H_o}{(\gamma+k)+(\gamma-k)z^{-1}} \prod_{k=1}^{N-1/2} \frac{1}{1+a_k\, z^{-1}}, \qquad (13)$$

where z_i – transfer poles, defined by expression (11).

The right part of the expression (13) is a product of transfer functions of *computational-effective bilines*, which full structures [1] allow to realize passbands on some orders less in comparison with biquades. The so-called ***truncated*** bilines, which transfer functions are included in expression (13), have the structure shown on Fig. 3.

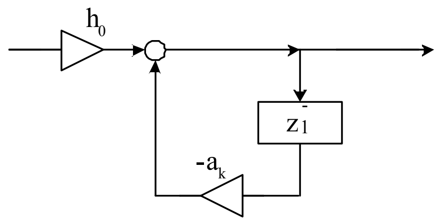


Figure 3. *Computational-effective (truncated) biline structure*

The clear advantage of such structures in comparison with traditional bilines consists in practically double simplification of their realization.

At high-frequency bilinear Z-transformation the ***cotangent frequency transformation*** is used. In this case it is possible to approximate cotangent by the cosine of the same argument. Characteristics of the narrow-band filters practically do not differ from ideal near to Nuiquist frequency (Fig. 7).

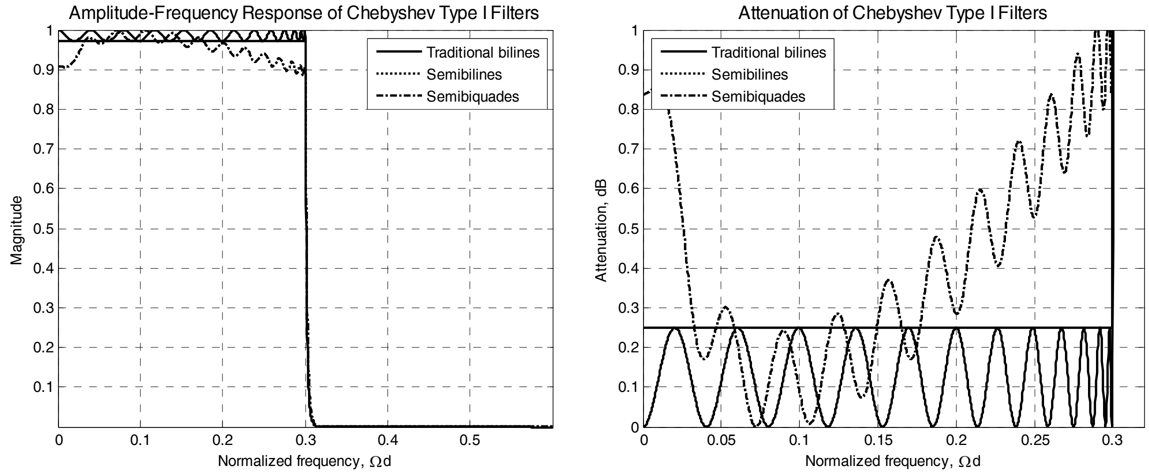
2.1. Examples of Low and High-Frequency Modified Structures

Let's show characteristics of the synthesized digital filters on computational effective bilines. For low-frequency sinus transformation it is better to use an area of digital normalized frequencies below 0.1. We shall set the following requirements to the filter:

1. The order of filter N = 25;
2. Passband ripple a_{max} = 0.25 dB.

On Fig. 4 the amplitude-frequency characteristics for two cut-off frequencies of transfer functions are shown:

$\widetilde{\Omega}_1 = 0.3 \qquad \widetilde{\Omega}_{control} = 1.5 \cdot \widetilde{\Omega}_1$



$\widetilde{\Omega}_1 = 0.1$

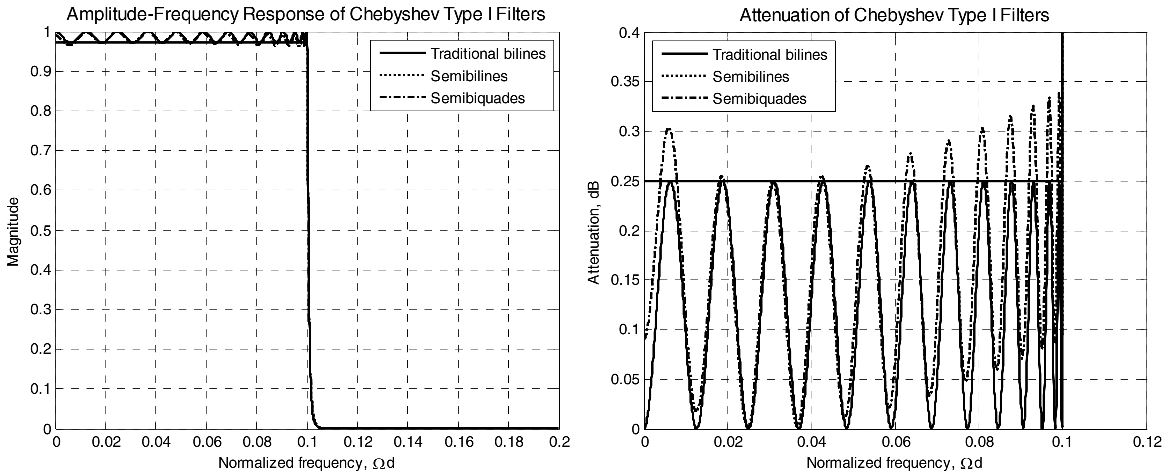


Figure 4. *Amplitude-frequency characteristics of Chebyshev's type I filter, realized on traditional and truncated structures*

On Fig. 5 the amplitude-frequency characteristics for the next cut-off frequencies of transfer function are shown:

$\widetilde{\Omega}_1 = 1 \cdot 10^{-5} \qquad \widetilde{\Omega}_{control} = 1.5 \cdot \widetilde{\Omega}_1$

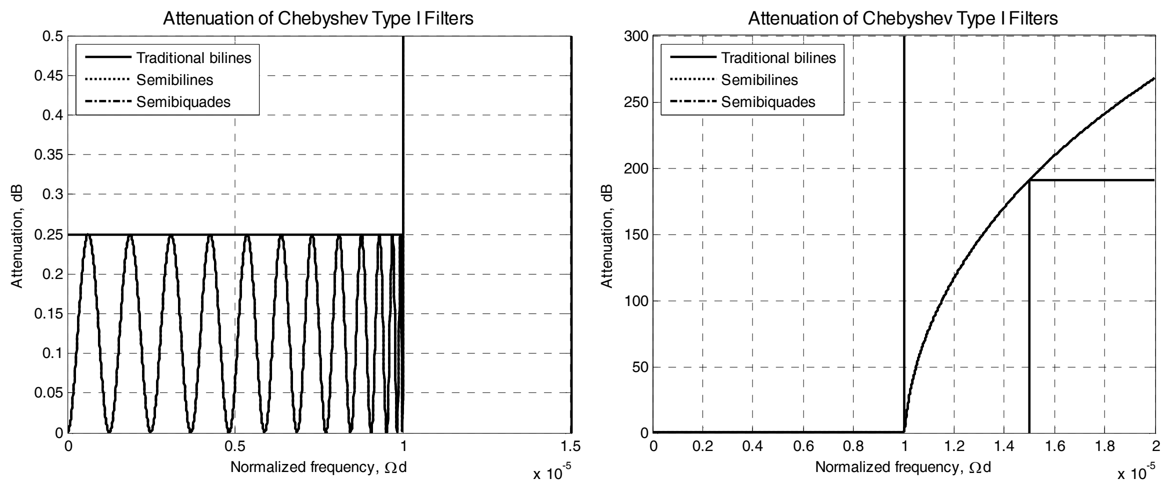


Figure 5. *Amplitude-frequency characteristics of Chebyshev's type I filter, realized on traditional and truncated structures*

On Fig. 6 the amplitude-frequency characteristics for cut-off frequency in the field of which and below AFR-distortions start are shown:

$$\tilde{\Omega}_1 = 1 \cdot 10^{-7} \qquad \tilde{\Omega}_{control} = 1.5 \cdot \tilde{\Omega}_1$$

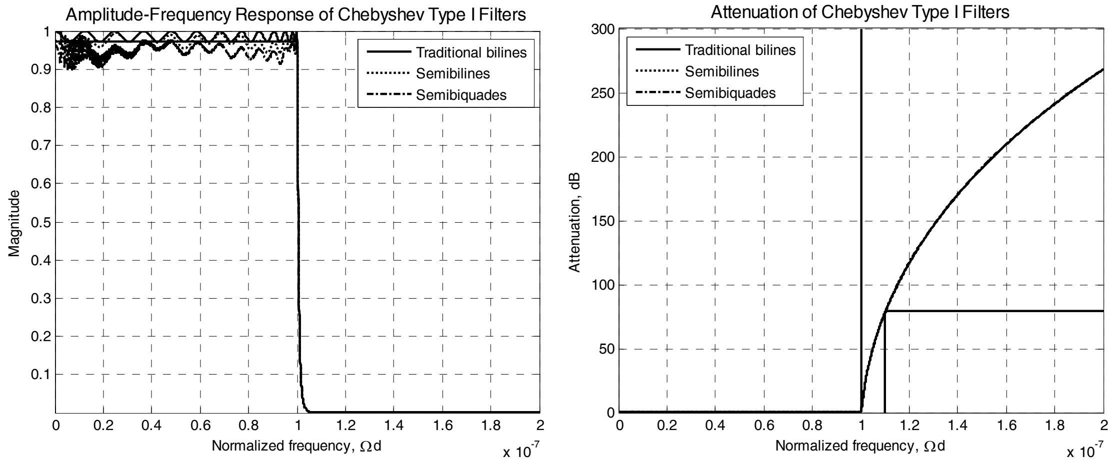


Figure 6. *Amplitude-frequency responses of synthesized structure in the area of critical cut-off frequencies*

As it is shown, attenuation on control frequency is completely satisfied to the set of requirements.

Analyzing frequency characteristics of the received new HF-structures, it is possible to show, that qualitatively they are identical to the characteristics shown above for LF case. Below requirements to the digital HF-filter and its frequency characteristics are shown.

$$N = 24 \qquad \tilde{\Omega}_1 = 0.9999997 \qquad \tilde{\Omega}_{control} = 0.99999967$$

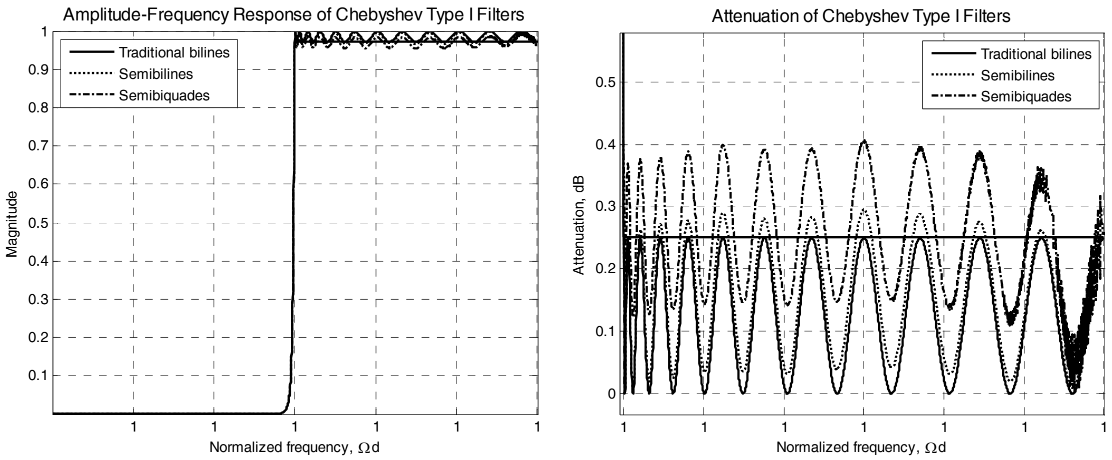


Figure 7. *Amplitude-frequency responses of synthesized structure in the area of critical cut-off frequencies*

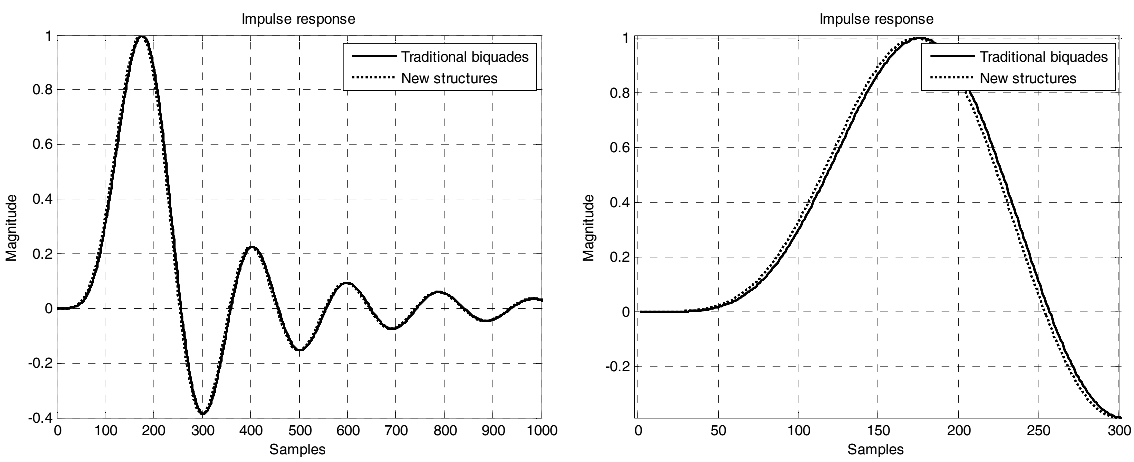


Figure 8. *Impulse responses of synthesized structure in the area of critical cut-off frequencies*

3. DESCRIPTION OF RESEARCH. PASSBAND MODIFIED STRUCTURES

Let's consider an opportunity of passband filters realization on computational effective structures. As it has been described above, while designing the new LF and HF-structures were used such Z-transformations, which at transition from the prototype to the digital filter kept transfer function in the form of a constant, divided on a polynomial. In case of the passband filter it is inconvenient to use simple transformation with the same properties, providing satisfaction of the frequency characteristics to the requirements in the field of supernarrow bands. Therefore we shall consider the following approach.

While using the standard passband Z-transformation the transfer function of N-order digital filter is the relation of two N-degree polynomials on z^{-1}. Generally zeros of transfer or poles of attenuation are provided with a transfer function's enumerator. It is known, that Chebyshev's type I filter has no poles of attenuation. Zeros of attenuation or satisfactions to the requirements in a passband are provided with a denominator of transfer function.

Let's consider the possibility of passband filter realization, which transfer function in Z-plane is described by the expression (14):

$$H(z^{-1}) = H_0 \prod_{i=1}^{M} B_{2i}(z^{-1}) \prod_{i=1}^{N/2} \frac{\dot{h}_{0i}}{A_{2i}(z^{-1})}, \tag{14}$$

where in common case the polynomial orders N ≥ 2M.

Suppose that the nominator of this transfer function is the constant (without first production) and the transfer function is described by the next expression:

$$H(z^{-1}) = H_0 \prod_{i=1}^{N/2} \frac{\dot{h}_{0i}}{1 + a_{1i} z^{-1} + a_{2i} z^{-2}}, \tag{15}$$

where coefficients a_i are the same to the defined above in expression (4). However, in this case the new frequency characteristics will not meet to the requirements, set in a passband.

So, in the case of expression (14) the denominator of initial transfer function is kept, and m 2[nd]-order polynomials are realized in the numerator; each of them provides one pole of attenuation outside of passband of the projected filter. The transfer function of the 2[nd]-order unit, providing a pole of attenuation on frequency $\tilde{\Omega}_i$, is described by the expression (16):

$$B_{2i}(z^{-1}) = 1 + 2b_{1i} z^{-1} + z^{-2} = 2z^{-1}\left(b_{1i} + \frac{z + z^{-1}}{2}\right) = 2z^{-1}\left(b_{1i} + \cos\left(\pi\tilde{\Omega}_i\right)\right),$$
$$b_{1i} = -\cos\left(\pi\tilde{\Omega}_i\right). \tag{16}$$

It is possible to include into transfer function as much the 2[nd]-order units, as it is needed, but in the simplest case let's show the example with only two attenuation poles outside the passband on the each sides of stopband.

Therefore we shall address to numerical methods and let's optimize the transfer function (14), so it would look as follows:

$$H(z^{-1}) = B_{21} B_{22} H_0 \prod_{i=1}^{N/2} \frac{\dot{h}_{0i}}{1 + a_{1i} z^{-1} + a_{2i} z^{-2}} \quad \Rightarrow$$

$$H(z^{-1}) = B_{21} B_{22} H_0 \prod_{i=1}^{N/2} \frac{\dot{h}_{0i}}{1 + (a_{1i} + \alpha_{1i} + j\beta_{1i}) z^{-1} + (a_{2i} + \alpha_{2i} + j\beta_{2i}) z^{-2}}. \tag{17}$$

As it was mentioned above, this transfer function may not satisfy to the requirements in the passband, if as initial approach the coefficients (a_1, a_2) of a of transfer function's denominator of the usual passband filter are used. So we should consider the next way to solve this problem. If we have a required a_{max} in the passband of the projected filter, so let's use another initial approach – coefficients (a_1, a_2) should be evaluated for the filter with $a_{max2} < a_{max1}$. So the suggested iteration procedure of the computational-filter synthesis should include the next steps:

1. Definition of the frequency characteristics requirements: a_{max}, a_{min}, cut-off and control frequencies.
2. Setting initial approach (a_1, a_2) as for filter, which passband ripple of its characteristic is less, than required a_{max}.
3. Definition of two poles of attenuation outside the passband on the both sides of stopband to make the frequency characteristics satisfy to the requirements on the control frequency (in fact, b_1 and b_2).
4. Optimization the transfer function of the expression (17), while the attenuation in the passband is equal or less to the required a_{max} and the requirements on the control frequency are completely satisfied.
5. If it's difficult or impossible to optimize passband ripple, another initial approach can be used (for example, evaluated based on the decreased a_{max}).

Selecting coefficients α and β, we optimize amplitude-frequency response by a numerical method (for example, Nelder-Mead).

3.1. Examples of Passband Modified Filters

Further, we shall show frequency characteristics of the passband filter, designed on a method, described above. We shall set the following requirements to the filter:

1. Order of the filter's prototype N = 12;
2. Passband ripple $a_{max} = 0.1$ dB;
3. Central frequency $\tilde{\Omega}_0 = 0.19$
4. Cut-off frequencies are $\tilde{\Omega}_{-1} = 0.189$ and $\tilde{\Omega}_1 = 0.191$.
5. Control frequencies are $\tilde{\Omega}_{-1} = 0.1885$ and $\tilde{\Omega}_1 = 0.1915$.

As it is shown on Fig. 9, if a denominator in the transfer function of the projected digital filter will remain without changes, the frequency characteristic would not meet to the set of requirements, even within the limits of admissible distortions.

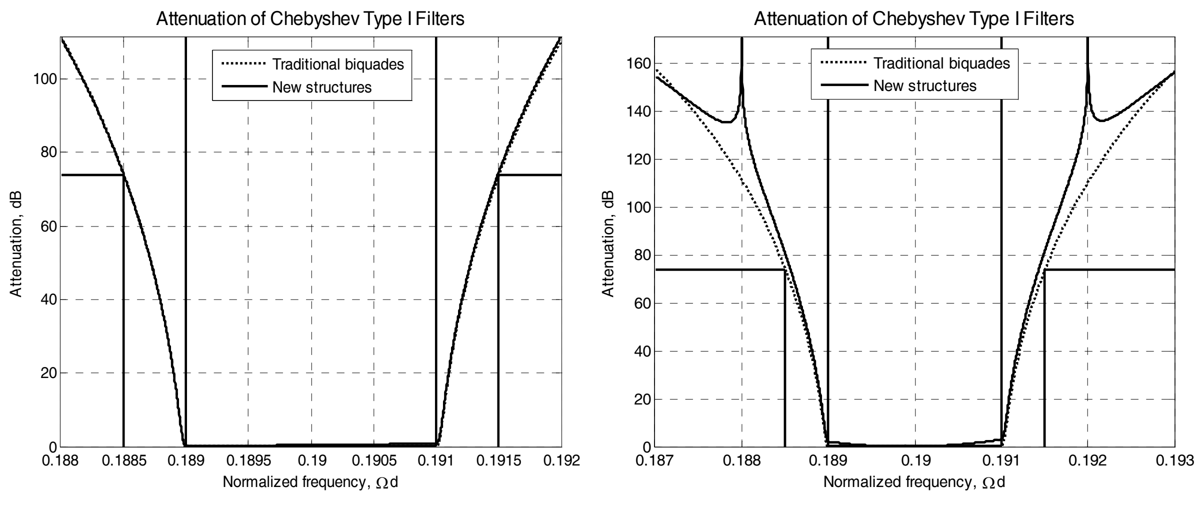


Figure 9. *Amplitude-frequency characteristics of synthesized passband filter without optimization*

Let's lead optimization by one of numerical methods, for example, by Nelder-Mead.

Before optimization we should evaluate initial approach – coefficients (a_1, a_2). As was explained above, this approach should be determined for the filter with the decreased a_{max}. So, suppose a_{max} = 0.04 dB. Then we need to set two poles of attenuation in the stopband area (Ω_{-1}, Ω_1). Selecting coefficients α_i and β_i in the transfer function described by the expression (16), we optimize amplitude-frequency response so, that its degree of a passband deviation would be the minimum, until it becomes equal or less than required a_{max} (0.1 dB).

The frequency characteristics of the optimized transfer function of simplified passband filter are shown on Fig. 10. Thus the maximal non-uniformity of passband attenuation is 0.09 dB in comparison with demanded 0.1 dB. Apparently on the figure, attenuation on the control frequency has reserve and some dB more, than provided by traditional biquad structure.

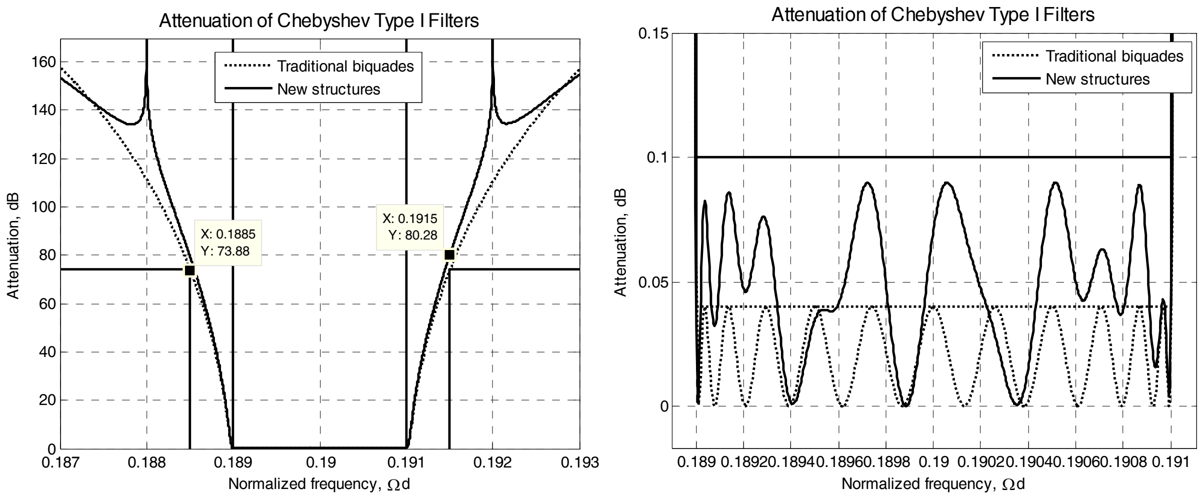


Figure 10. *Amplitude-frequency characteristics of synthesized passband filter after coefficients' optimization*

Let's reduce the passband up to $2 \cdot 10^{-4}$ and repeat the AFR optimization until the attenuation in a passband is equal to 0.09 dB. Results are shown on Fig. 11.

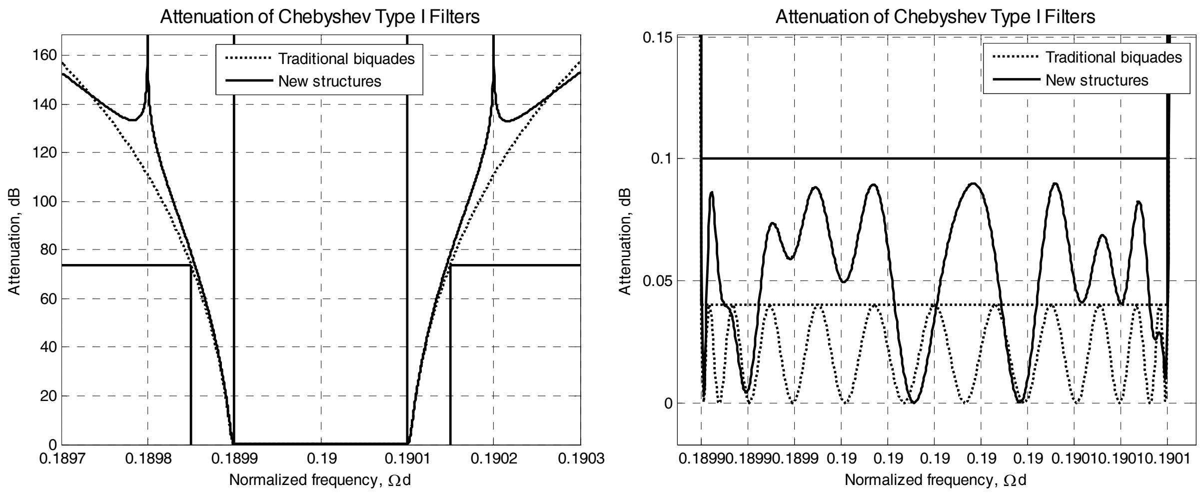


Figure 11. *Amplitude-frequency characteristics of synthesized passband filter after coefficients' optimization*

Let’s arrange 2 poles on zero frequency and Nuiquist frequencies that are on 0 and 1 of normalized scale. We optimize the AFR in a passband at all other requirements. Results are shown on Fig. 12.

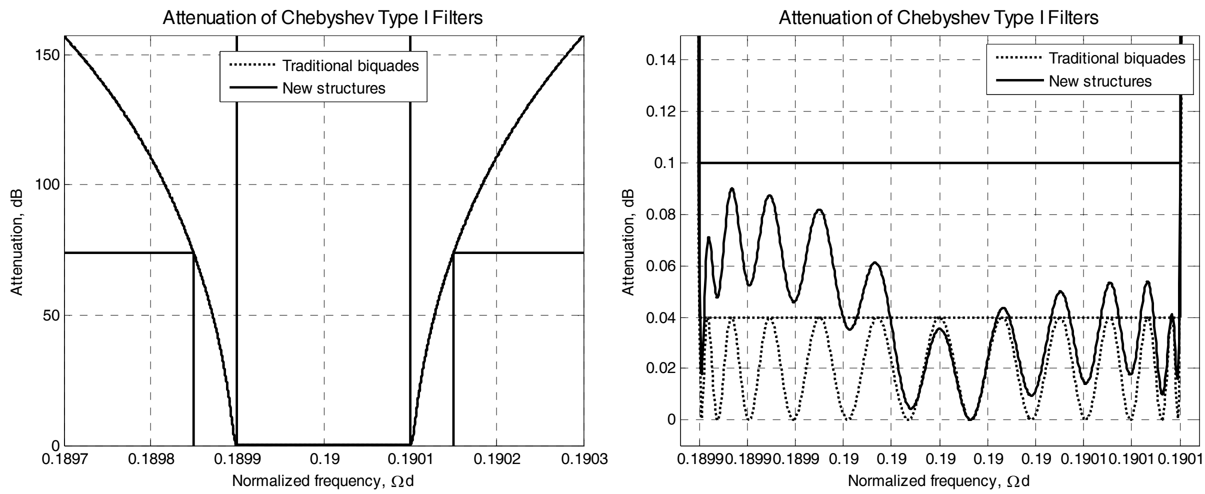


Figure 12. *Frequency characteristics of the traditional biquad filter and "truncated" structure with poles at edges of a frequency range*

As it is shown on Fig. above, at the set passband ripple 0.1 dB the maximal deviation from requirements makes nearby 0.05 dB. Moreover, the characteristics of the optimized structures completely meet the requirements on control frequency and inside the passband.

Let's estimate phase characteristics and group delay of the synthesized structures. On Fig. 13 phase characteristics of the passband filter, realized on traditional biquades and computational-effective structures, optimized by AFR are shown.

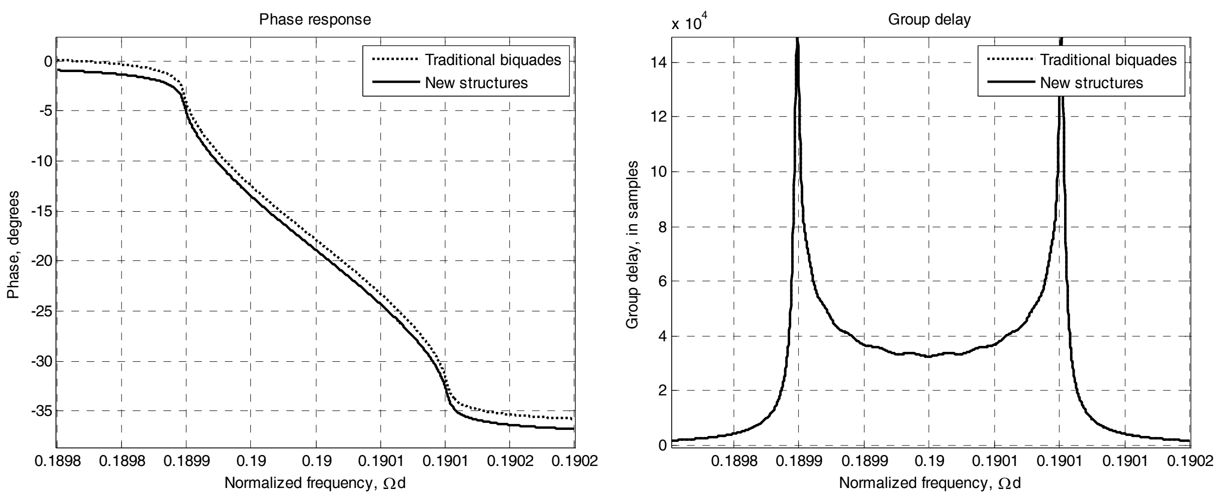


Figure 13. *Phase characteristics and group delays of the traditional biquad filter and "truncated" structure with poles at edges of a frequency range*

As it is shown on Fig. 13 (b), the group delay of modified simplified structures is almost the same as usual.

All transfer function’s poles of the optimized structures are arranged inside the unit circle, as it is shown on Fig. 14. Thus the time stability and possibility to realize these filters in real time are provided.

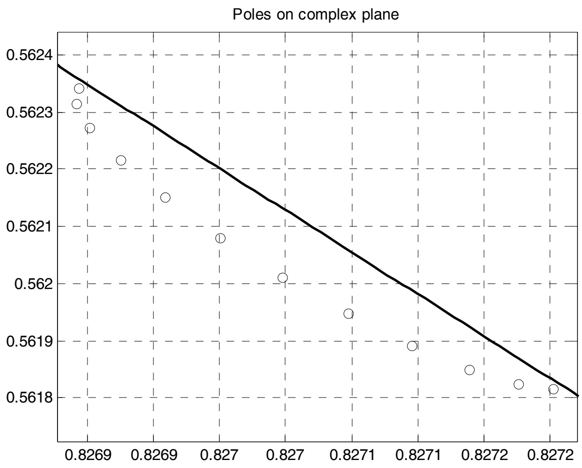


Figure 14. *Poles of the synthesized structures on the complex plane*

4. CONCLUSIONS

The new class of polynomial characteristics is offered. It is shown, that modified LF and HF-filters possess qualitatively almost the same frequency characteristics in comparison with traditional biquad structures with the same set of requirements.

Method and examples of realization of passband digital filters on computational-effective biquades and bilines are shown. Some advantages before traditional Chebyshev's type I biquad filters are shown.

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